Theoretical Physics
Blackett Laboratory
IMPERIAL COLLEGE LONDON

OCTONIONIC ASPECTS OF SUPERGRAVITY

Alexander Close
Supervised by Professor Michael Duff

Submitted in partial fulfilment of the requirements for the degree of
Master of Science of Imperial College London

20 September 2013

“The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.”

Eugene Wigner

Acknowledgements

I wish to thank Mike Duff for allowing me to undertake an M.Sc. project in such an interesting area, and to his team, particularly Leo Hughes, for making sources and literature available, and for being there for advice.

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TOWARD UNIFICATION

The principle of unification is not driven by any *a priori* reasoning, but by the pursuit of an aesthetic quality, that the universe should in some sense be simple and elegant. However unfounded one may find this supposition, it has proven incredibly useful as a driving principle of theoretical physics. The first unification is widely cited as Maxwell electromagnetism, though it could be argued to have occurred before this when thermodynamics met statistical physics, or even when Newton unified physics and astronomy. In the context of particle physics the principle is a continuation of Maxwell, but concerning a unified field theory treatment of the known fundamental interactions.

To incorporate gravity with the other forces of the Standard Model one first needs to formulate gravity as a quantum theory, an extremely problematic reconciliation in terms of renormalizability. But it has been shown that the quantum theory of interacting spin-2 bosons inevitably leads to the emergence of a structure indistinguishable from general relativity [9, 16]. Such mediating particles are the hypothetical gravitons. While quantum field theory can be performed on curved spacetimes, leading to such ideas as Hawking radiation, it is still only a semi-classical theory of quantum gravity. That there is such a disconnect between the two pillars of modern theoretical physics seems a boding of a new scientific revolution, where either general relativity or quantum mechanics (or both) will have to be significantly modified or altogether replaced by a more encompassing theory. There are various and varied lines of inquiry from which the appropriate step might be identified upon historical reflection, but one promising venture is that of supergravity. It involves conjecturing a new but unseen particle symmetry.

1.1 Wherefore 'superstuff'?

Supersymmetry (SuSy) is an ambitiously proposed symmetry relating the two "types" of fundamental particle, bosons (force particles) and fermions (matter particles), and requires that the existence of any particle species implies the existence of one of the opposite type, its superpartner. An aesthetically promising extension of Poincaré spacetime symmetry, it unifies the description of these two basic particles (which were previously separate entities with separate spin statistics theorems) under a single mathematical framework, side-stepping the Coleman-Mandula theorem, cancelling loop divergences that plague particle theories, allowing gauge coupling unification and even providing a speculative candidate for so-called "dark matter". The SuSy generators, being a generalization of those of Poincaré symmetry, already invoke the structure of spacetime in the transformations and update special relativity to make it quantum mechanical. A remarkable feature is that when one promotes SuSy to a local gauge symmetry, gravity is automatically incorporated when a spin-2 second-rank tensor boson emerges with its spin-3/2 superpartner, the gravitino, such that one has a supersymmetric extension of general relativity: supergravity. SuSy is, in addition to all of this, at the time of writing and probably to the chagrin of many, an unobserved symmetry even in its broken form. But although hypothetical to physicists, interest in SuSy is alive in pure mathematics. For there exists an emerging picture of connectedness between various mathematical branches, bridged by the study of supersymmetric concepts that are interesting in their own right. Its study has grown, encroaching into other disciplines and prepending "super" to foreign terminology.

SuSy is a necessary stepping stone to the best current candidate "theory of everything", the overarching *theoria incognita*, M-theory [35]. If this somewhat contentious [11] proposal is true then 11-dimensional supergravity must be its low-energy limit. Though supergravity itself does not necessitate string or M-theory, it seems these are needed because of a fatal flaw: since it is most simple in 11 dimensions, and its ultraviolet completion, the various superstring theories, which live in 10 dimensions (and are various limits of M-theory), there must be reconciliation with observation in that various means of dimensional compactification must be invoked (the different ways this can be done causing differing lower-dimensional physical outcomes)—but conventional (manifold) compactification cannot derive "handedness" from the theory, which is a key property of Nature. (It is, however, possible to derive handedness when the space of compactification is singular.) Historically, string theories displaced supergravity for this and other reasons until in 1995 Edward Witten proposed that they are all facets of M-theory.

1.2 Algebras in particle physics

Developments in theoretical and mathematical physics have pushed for a greater alegebrization and geometrization of particle physics [17]. This is the case not just for the well-established particle theories of the last century but also for modern research. These developments are spurred by the demonstrable power of symmetries, with supersymmetry itself responsible for the development and introduction into physics of, respectively, the super-Poincaré and Grassmann algebras. SuSy then lends itself to supergravity and superstrings, with their own symmetries and dualities. The octonions come into play here since their existence is tied up with the exceptional Lie and exceptional Jordan algebras. There seems to be many secrets hidden in such mathematically murky structures. The octonions should then have something to say about physical theories whose symmetries incorporate them.

The Standard Model contains chiral fermions with handedness that transform differently under the electroweak gauge symmetry, meaning that the irreducible representations are Weyl and not Dirac spinors. These can only be projected out by projection operators built with the imaginary unit i, showing that Nature has a preference for the complex numbers. Indeed, quantum mechanics is built on complex Hilbert spaces, with the unit quaternions (being isomorphic to the SU(2) group) providing a simpler environment in which to study spin. But octonionic quantum mechanics has met with little success.

However, in addition to their appearance in the work documented here, the octonions along with supergravity have recently found a place for making predictions in areas such as quantum information theory [10, 23]. Some authors have even suggested that their non-associativity may be the underlying cause of colour confinement [17]. However, for now it is pertinent to describe their ultimate relevance in physics, and hence the 'real' world, as remaining cloaked in mystery. The octonions are 8-dimensional, non-associative numbers, meaning $x(yx) \neq (xy)z$. After having been kept off the radar for most of the period since their inception due to this latter characteristic, there has been a renewed interest in their application for theories such as string theory. It seems natural that such objects could have a role in describing higher-dimensional spacetimes. Superstrings and branes are extended objects and require at least 10 dimensions in which to propagate. Supergravity requires a minimum of 11 in order to embed the gauge groups of the Standard Model [34]. Could formulating such theories over the octonions provide a way to bring them back to the familiar 4-dimensional world?

1.3 A notational note

Throughout this dissertation the following notational rules will apply. Lowercase indices from the middle of the Greek alphabet such as μ, ν will denote the components of any full division algebra, while lowercase Latin indices of the range i, j, k will be used to denote the algebra's imaginary subspace. Thus one has $x^\mu e_\mu \in \mathbb{O}$ and $q^i e_i \in \mathrm{Im}\,\mathbb{H}$. Lorentz spacetime indices will be written using lowercase letters from the middle of the Latin alphabet, such as in g_{mn}. Less importantly, lowercase Greek indices from the beginning of the alphabet will mean spinor components. For instance, a gravitino which has both spinor and 1-form components will be denoted ξ^α_m, though for general spinors this index will be mostly suppressed. Generic non-division algebras are written in Gothic lettering, with subscripts marking rank. So, unlike some sources, the algebra of 3×3 antihermitian matrices over field $\mathbb{F}$ will here be expressed as $\mathfrak{a}(3, \mathbb{F})$, and not $\mathfrak{a}_3$, which is the Lie algebra of the group $\mathrm{SU}(4)$. A representation of a rank-r algebra will be written in Dynkin notation as r weights between square brackets. As an example, the adjoint of $\mathfrak{so}(8)$ is written [0100].

PRELIMINARY IDEAS

A foundational acquaintance with group and representation theory, supersymmetry (SuSy), supergravity (SuGra) and super Yang-Mills (sYM) theory will be assumed throughout.

2.1 Normed division algebras

Define an algebra A as a (real) vector space that is also equipped with a bilinear map $m: A \times A \to A$ under which the vector space is closed. This map may be called 'multiplication': $m(x, y) = xy$. The algebra is a division algebra if for any $x, y \in A$ and $xy = 0$ then either $x = 0$ or $y = 0$. That is, left or right multiplication by any nonzero element has an inverse operation, giving the name 'division'. If the vector space is a normed vector space, and $\|x\|\|y\| = \|xy\|$, then A is a normed division algebra (NDA), which here will be promoted notationally to $\mathbb{A}$.

There are precisely four [18] NDAs: the reals, complexes, quaternions and octonions; or $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, respectively. The reals are the only algebra of the four to be ordered. Moving from the real line to the complex plane, this property is lost. Likewise, the quaternions lose the property of commutativity that both $\mathbb{R}$ and $\mathbb{C}$ have, yet they remain associative. The octionions, however, further lose this property and are non-associative. For this reason a further trilinear map, $a: \mathbb{A} \times \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, whereby

$$[x, y, z] \to (xy)z - x(yz) \tag{2.1}$$

called the associator can be defined. Since it measures the failure of associativity, it is zero for all the NDAs save the octonions. Because the NDAs are also alternating algebras [39], the sign of the associator and commutator flips when any two elements are switched.

The basis of $\mathbb{R}$ is $\{1\}$. If $i = \sqrt{-1}$ is used in addition to define the basis $\{1, i\}$, $\mathbb{C}$ can be built using a pair of elements a, b from $\mathbb{R}$ such that a generic element of $\mathbb{C}$ is $z = a + ib$. Two elements of the one-dimensional algebra $\mathbb{R}$ are thus needed to specify any element of $\mathbb{C}$, making the latter a two-dimensional algebra. Now taking two additional bases j, k such that $i^2 = j^2 = k^2 = ijk = -1$ then a pair of elements from $\mathbb{C}$ can be used to construct a generic element of the four-dimensional quaternion algebra $\mathbb{H}$:

$$
\begin{aligned}
q &= (a + ib) + k(c + id) \\
&= a + ib + jd + kc \qquad \in \mathbb{H}
\end{aligned}
$$

where $a, b, c, d \in \mathbb{R}$. It is clear from this construction that the quaternions are in general non-commutative. The next step is to take two elements of $\mathbb{H}$ and use them to build an octonion. The basis of $\mathbb{O}$ thus has one real element, $e_0 = 1$, and seven imaginary ones, e_i, which together as e_μ ($\mu = 0, \ldots, 7$) span an 8-dimensional vector space. Any $x \in \mathbb{O}$ can then be written $x = x^\mu e_\mu = x^0 + x^i e_i$ ($i = 1, \ldots, 7$). Elements of $\mathbb{O}$ and $\mathbb{H}$ are conjugated analogously to the complex numbers: by reversing the sign of the imaginary components: $x^* = x_0 - x^i e_i$. Hence the ordered property of $\mathbb{R}$ translates to self-conjugacy. Conjugation can then be used to define the norm and inner product (which is just the Euclidean inner product inherited from $\mathbb{R}^n$, $n = \dim\mathbb{A}$) on $\mathbb{A}$.

Constructing NDAs in this fashion, each successively doubling in dimension, is known as the Cayley-Dixon procedure. Clearly this process could be repeated *ad infinitum* to produce an endless supply of algebras. But other than $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ none of these will be division algebras.

The octonionic multiplication rule is given by:

$$
\begin{aligned}
e_\mu e_\nu &= (\delta_{\mu 0}\delta_{\nu\rho} + \delta_{0\nu}\delta_{\mu\rho} - \delta_{\mu\nu}\delta_{0\rho} + L_{\mu\nu\rho})e_\rho \\
&= \Gamma^\mu_{\nu\rho} e_\rho
\end{aligned} \tag{2.2}
$$

where $\delta_{\mu\nu}$ is the 8-dimensional Euclidean metric. $L_{\mu\nu\rho}$ is a totally antisymmetric object which is nonzero only for $\mu\nu\rho = ijk$. Hence $L_{0\mu\nu} = 0$. The Fano plane (Figure 2.1) provides

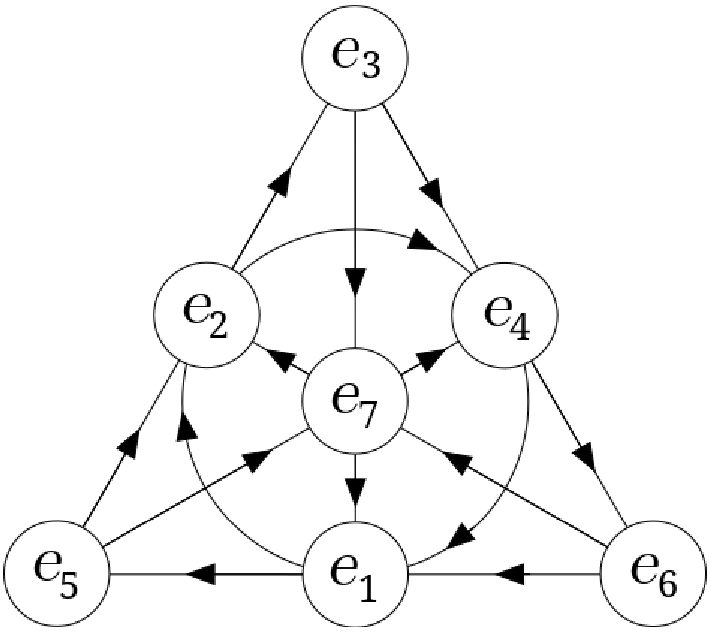

Figure 2.1: The Fano plane provides the multiplication rule for imaginary elements of the octonionic basis $e_i \in \mathbb{Z}_7$.

a visualization of L_{ijk} in terms of a structure with seven points and seven 'lines', each point representing a label of the imaginary bases. Clearly any three points on a line give the multiplication rule of the quaternion subalgebra (where L_{ijk} in (2.2) becomes ϵ_{ijk}). In general, if three octonion bases generate a quaternion subalgebra then they will associate. Otherwise, they will not. As an explicit example,

$$e_1(e_2e_3) = e_1e_5 = e_6 = -(-e_6) = -(e_4e_3) = -(e_1e_2)e_3$$

In addition, any quadrangle of the Fano plane (arrived at by removing a line) represents another totally antisymmetric and purely vectorial structure tensor $Q_{\mu\nu\rho\sigma}$ that determines the associator of the octonions:

$$[e_\mu, e_\nu, e_\rho] = 2Q_{\mu\nu\rho\sigma}e_\sigma \tag{2.3}$$

where again only Q_{ijkl} is nonzero. For i, j, k, l taking the values of an allowed quadrangle, $Q_{ijkl} = -1$. The tensors L_{ijk} and Q_{ijkl} are dual to each other in $\mathbb{R}^7$—the vector space spanned by the imaginary octonions [12].

2.2 The octonions and SO(n)

The vector space spanned by the octonionic basis is 8-dimensional. There actually exists an intimate connection with the representations of the 28- and 21-dimensional special orthogonal groups SO(8) and SO(7) via the octonion structure tensors. The object $\Gamma^{\mu}_{\nu\rho}$ was defined in (2.2). Define also

$$\bar{\Gamma}^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu} \tag{2.4}$$

such that

$$e_{\mu}e_{\nu} = \Gamma^{\mu}_{\nu\rho}e_{\rho} \tag{2.5}$$

$$e^{*}_{\mu}e_{\nu} = \bar{\Gamma}^{\mu}_{\nu\rho}e_{\rho}. \tag{2.6}$$

The symbol Γ has been chosen to anticipate the fact that these objects in fact satisfy

$$\Gamma^{\mu}\bar{\Gamma}^{\nu} + \Gamma^{\nu}\bar{\Gamma}^{\mu} = \bar{\Gamma}^{\mu}\Gamma^{\nu} + \bar{\Gamma}^{\nu}\Gamma^{\mu} = 2\delta_{\mu\nu}\mathbb{1} \tag{2.7}$$

which is the Spin(8) Clifford algebra. Thus one can build the antisymmetric objects

$$\Sigma^{[\mu\nu]} = \frac{1}{2}\Gamma^{[\mu}\bar{\Gamma}^{\nu]} \qquad \bar{\Sigma}^{[\mu\nu]} = \frac{1}{2}\bar{\Gamma}^{[\mu}\Gamma^{\nu]} \tag{2.8}$$

which respectively generate the spinor and conjugate spinor representations, $\mathbf{8}_s$ and $\mathbf{8}_c$, of SO(8) from octonion products.

In the same way that the quaternions can generate 3-dimensional geometry through their imaginary parts, the product of an octonion and imaginary octonion provides for SO(7) geometry.

$$e_i e_{\mu} = \delta_{0\mu}\delta_{i\nu} - \delta_{i\mu}\delta_{0\nu} + L_{i\mu\nu} \tag{2.9}$$

$$= \Gamma^{i}_{\mu\nu}e_{\nu} \tag{2.10}$$

with $\Gamma^{i}_{\mu\nu}$ satisfying the Spin(7) Clifford algebra. It is known that for SO(n = odd) there is just one spinor representation. This is realized here by the fact that $\Gamma^{i}_{\mu\nu} = -\Gamma^{i}_{\nu\mu}$, so the spinor representations generated by $e_i e_{\mu}$ and $e^{*}_{i} e_{\mu} = -e_i e_{\mu}$ are actually the same. Then,

$$\Sigma^{[ij]} = \frac{1}{2}\Gamma^{[i}\Gamma^{j]} \tag{2.11}$$

generates the **8** of SO(7). The vector representations are generated simply by the metric inherited from $\mathbb{R}^n$.

$$J_{[\mu\nu]\rho\sigma} = \delta_{\rho\mu}\delta_{\nu\sigma} - \delta_{\rho\nu}\delta_{\mu\sigma} \tag{2.12}$$

generates the $\mathbf{8}_v$ of SO(8), while

$$J_{[ij]kl} = \delta_{ki}\delta_{jl} - \delta_{kj}\delta_{il} \tag{2.13}$$

generates the **7** of SO(7). From a geometric perspective the octonions do not seem to be able to generate a large enough group of rotations to get the full SO(7). The full group is however generated due to the nonassociativity of the algebra: two conjugations are ill-defined without brackets, so there is an increase in the number of rotations available.

2.3 NDAs and Minkowski spacetime

There is a known isomorphism between the Lie algebras of the groups SO(1,3)—the proper Lorentz group—and SL(2, $\mathbb{C}$). In fact, this is one case of a more general Lie algebra isomorphism [30]:

$$\mathfrak{so}(1, n+1) \cong \mathfrak{sl}(2, \mathbb{A}) \tag{2.14}$$

with n the dimension of $\mathbb{A}$. Because of this, Lorentz spacetimes of dimensions 3,4,6 and 10 can be formulated in terms of 2×2 matrices with entries in the corresponding NDA. In the massless case the Little group is considered: SO$(1, n+1)$ is reduced to SO(n). In all cases there is a simple generalization to SO$(1, n+2)$.

In the case of 4-dimensional Minkowski spacetime, the spinor and conjugate spinor representations are irreducible Weyl spinors that transform according to the infinitesimal relation:

$$\begin{aligned} \delta\Psi_{\mathbb{C}} &= \frac{1}{4}\lambda^{mn}\sigma_{mn}\Psi_{\mathbb{C}}, \\ \delta X_{\mathbb{C}} &= \frac{1}{4}\lambda^{mn}\bar{\sigma}_{mn}X_{\mathbb{C}}. \end{aligned} \tag{2.15}$$

The notation $F_{\mathbb{A}}$ means the field F valued in $\mathbb{A}$ and the Latin letters denote spacetime indices that run over $m = 0, \ldots, n+1$. In the 4-dimensional case then, the 2×1 Weyl spinors transform by left-action of a 2×2 matrix with entries in $\mathbb{C}$. The Pauli matrices act as a basis, with $\sigma_{mn} = \sigma_m\bar{\sigma}_n$ and $\bar{\sigma}_{mn} = \bar{\sigma}_m\sigma_n$. There are four Pauli matrices for the 4-dimensional case. If the one non-real Pauli matrix, σ_2, is removed then the remaining three real 2×2 matrices that act as a basis for the 3-dimensional $\mathbb{A} = \mathbb{R}$ case. In the full generalization, one has:

$$\sigma_0 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_{\mu+1} = \begin{bmatrix} 0 & e_\mu^* \\ e_\mu & 0 \end{bmatrix}, \qquad \sigma_{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.16}$$

The spinor transformations in (2.15) are not general enough to handle octonionic multiplication since they are ambiguous with regards to non-associative objects. The consistent choice is to define the order of operation from right to left. Thus

$$\begin{aligned} \delta\Psi_{\mathbb{O}} &= \frac{1}{4}\lambda^{mn}\sigma_m(\bar{\sigma}_n\Psi_{\mathbb{O}}), \\ \delta X_{\mathbb{O}} &= \frac{1}{4}\lambda^{mn}\bar{\sigma}_m(\sigma_n X_{\mathbb{O}}). \end{aligned} \tag{2.17}$$

The algebra of 2×2 Hermitian matrices $\mathfrak{h}(2,\mathbb{A})$ forms an $n+2$ vector space pertaining to the vector representation, again using the relevant Pauli basis.

$$V_{\mathbb{A}} = V^m\bar{\sigma}_m = \begin{bmatrix} V^0 + V^{n+1} & V^{\mu+1}e_\mu^* \\ V^{\mu+1}e_\mu & V^0 - V^{n+1} \end{bmatrix} \in \mathfrak{h}(2,\mathbb{A}) \tag{2.18}$$

For $V_{\mathbb{A}}$ to be Hermitian the diagonal elements are real. It is readily seen that the determinant

$$-\det(V) = -(V^0)^2 + (V^{\mu+1})^2 + (V^{n+1})^2 = V^m v_m = \eta(V_{\mathbb{A}}, V_{\mathbb{A}}) \tag{2.19}$$

gives the Minkowski norm, which is persevered under $\mathfrak{sl}(2,\mathbb{A})$. The inner product of two vectors $V, W \in \mathfrak{h}(2,\mathbb{A})$ can be written as $V_{\mathbb{A}}^m W_m = \mathrm{Re}\ \mathrm{tr}\big(V_{\mathbb{A}}\tilde{W}_{\mathbb{A}}\big)/2$ with

$$\tilde{V}_{\mathbb{A}} = -V^m\sigma_m = V_{\mathbb{A}} - \mathrm{tr}(V_{\mathbb{A}})\mathbb{1} \tag{2.20}$$

as the trace reversal. This is equivalent to time reversal, since $\tilde{} : V^0 \to -V^0$. The generalized vector transformation is

$$\delta V_{\mathbb{A}} = \frac{1}{4}\lambda^{mn}\Big(\sigma_m(\bar{\sigma}_n V_{\mathbb{A}}) - V_{\mathbb{A}}(\bar{\sigma}_m\sigma_n)\Big). \tag{2.21}$$

A remarkable consequence of the standard reduction of $SO(1, n+1)$ to its Little group $SO(n)$ in the massless case is that the above fields are parameterized by single elements of $\mathbb{A}$. To construct the Little group, dependence on the Pauli bases that do not correspond to e_μ is killed. This is achieved by setting

$$\lambda^{0m} = \lambda^{n+1,m} = 0 \tag{2.22}$$

and identifying

$$\lambda^{\mu+1,\nu+1} \equiv \theta^{\mu\nu} \tag{2.23}$$

with basis

$$\sigma_{\mu\nu} = \begin{bmatrix} e_\mu^* e_\nu & 0 \\ 0 & e_\mu e_\nu^* \end{bmatrix}. \tag{2.24}$$

Writing the Little fields in lower case letters, the transformations of the three fields in n dimensions become

$$\begin{aligned} \delta\psi_{\mathbb{A}} &= \frac{1}{4}\theta^{\mu\nu} e_\mu^*(e_\nu \psi_{\mathbb{A}}) \\ \delta\chi_{\mathbb{A}} &= \frac{1}{4}\theta^{\mu\nu} e_\mu(e_\nu^* \chi_{\mathbb{A}}) \\ \delta v_{\mathbb{A}} &= \frac{1}{4}\theta^{\mu\nu}\Big(e_\mu(e_\nu^* v_{\mathbb{A}}) - v_{\mathbb{A}}(e_\mu^* e_\nu)\Big). \end{aligned} \tag{2.25}$$

It can be seen that in the Little group the dimensions of the spinors and vector are equal and valued at n. Hence these fields are parameterized by single numbers in $\mathbb{A}$. In the case $\mathbb{A} = \mathbb{R}$ the Little group $SO(1) = 1$ is trivial and the field parameters vanish.

2.4 Jordan, triality and other algebras

Jordan algebras arose from the study of the algebra of Hermitian matrices. A Jordan algebra is a real vector space with commutative bilinear product that obeys the Jordan identity,

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x) \tag{2.26}$$

where

$$x \cdot y \equiv \frac{1}{2}(xy + yx)$$

is the Jordan product. This makes the algebra power associative. A non-associative algebra $\mathfrak{J}(n,\mathbb{A})$ of $n \times n$ Hermitian matrices over a field $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is a simple Jordan algebra. The only exceptional Jordan algebra is the case $\mathbb{A} = \mathbb{O}$, $n = 3$, and is thus 27-dimensional. This exceptional algebra is related not just to the octonions but to the exceptional Lie algebras. G_2 is the automorphism group of the octonions, while F_4 is the automorphism group of the exceptional Jordan algebra (their respective derivations on the algebra level). The remaining exceptional Lie algebras appear in Tits' construction of the magic square, which uses the Jordan algebras. The development of Jordan algebras brought along with it spin factors, which are not Hermitian matrices but nevertheless meet the Jordan axioms.

The Spin(8) group has three 8-dimensional irreducible representations (irreps): $\mathbf{8}_v, \mathbf{8}_s$ and $\mathbf{8}_c$. The Dynkin diagram for the algebra $\mathfrak{d}_4 \cong \mathfrak{spin}(8) \cong \mathfrak{so}(8)$ (Figure 2.2) possesses an S_3 (permutation) symmetry, the highest symmetry of any Dynkin diagram. This is

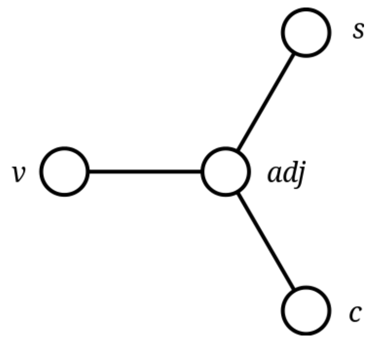

Figure 2.2: The Dynkin diagram for the D_4 Lie algebra with the vector, spinor, conjugate spinor and adjoint nodes labelled.

an example of a triality, a relationship between three vector spaces. A triality is a non-degenerate trilinear map,

$$t : V_1 \times V_2 \times V_3 \to \mathbb{F}$$

so that if one argument of the trilinear map is nonzero then it induces a duality between the other two. By choosing nonzero elements of the spaces such that they are isomorphic, the map can be recast as a bilinear multiplication:

$$m : V \times V \to V$$

which implies [4] V is a division algebra, and a normed division algebra when the field $\mathbb{F} = \mathbb{R}$. Conversely, the existence of any division algebra implies a triality, and NDAs imply normed trialities: $\| t(v_1, v_2, v_3)\| = \|v_1\|\|v_2\|\|v_3\|, v_n \in V_n$. Normed trialities therefore occur in dimensions 1, 2, 4 and 8. It is known that an n-dimensional NDA provides a representation of the Clifford group $Cl(n-1)$. There is in turn an isomorphism between $\mathrm{C}l(n-1)$ and $\mathrm{C}l_0(n)$, the subgroup of $\mathrm{C}l(n)$ constructed from all linear combinations of products of an even number of vectors in $\mathbb{R}^n$. Representations of the $\mathrm{Pin}(n)$ subgroup of $\mathrm{C}l(n)$ are pinors, while those restricted in $\mathrm{C}l_0(n)$ to $\mathrm{Spin}(n)$ are spinors. The NDAs then provide representations of $\mathrm{Spin}(n)$, as shown above. For $n = 8$, $\mathbb{O}$ provides a representation of Spin(8), the double cover of SO(8), whose Lie algebra $\mathfrak{d}_4$ is given by the D_4 Dynkin diagram.

The automorphism group of the triality is the subgroup of $\mathrm{SO}(n) \times \mathrm{SO}(n) \times \mathrm{SO}(n)$ which preserves the trilinear map, and the *triality algebra* is its Lie algebra:

$$\begin{gathered}\mathfrak{tri}\,\mathbb{A} \equiv \left\{(A,B,C) | A(xy) = (Bx)y + x(Cy)\right\} \\ A,B,C \in \mathfrak{so}(\mathbb{A}), \quad x,y \in \mathbb{A}\end{gathered} \tag{2.27}$$

where $\mathfrak{so}(\mathbb{A})$ is the norm-preserving algebra and is isomorphic to $\mathfrak{so}(n)$ (where $n = \dim\mathbb{A}$). It results [6] in the following.

$$\begin{aligned}&\mathfrak{tri}\,\mathbb{R} \cong \emptyset \\ &\mathfrak{tri}\,\mathbb{C} \cong \mathfrak{so}(2) \oplus \mathfrak{so}(2) \cong \mathfrak{u}(1)^2 \\ &\mathfrak{tri}\,\mathbb{H} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3) \cong \mathfrak{sp}(1)^3 \\ &\mathfrak{tri}\,\mathbb{O} \cong \mathfrak{so}(8).\end{aligned} \tag{2.28}$$

The derivation algebra $\mathfrak{der}(\mathbb{A})$ is a Lie algebra, a subspace of the associative algebra of all linear operators on $\mathbb{A}$. It provides another way of building the triality algebra:

$$\mathfrak{tri}\,\mathbb{A} \cong \mathfrak{der}\,\mathbb{A} \oplus \mathrm{Im}\,\mathbb{A}. \tag{2.29}$$

The nonzero $\mathfrak{der}\,\mathbb{A}$ are $\mathfrak{sp}(1)$ and $\mathfrak{g}_2$ for $\mathbb{H}$ and $\mathbb{O}$.

The automorphism groups of the NDAs themselves are for $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ respectively: 1, $\mathbb{Z}_2$, SO(3) and G_2. These are subgroups of the automorphism groups of the trialities.

$\mathbb{A}_1/\mathbb{A}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Table 2.1: The magic square of Lie algebras.

The existence of G_2 can then be accounted for as the automorphism group of the octonions from the theory of trialities and, by extension, the theory of spinors. Aligning the imaginary octonionic bases on the 7-sphere give $6+5+3=14=\dim G_2$ choices.

2.5 Freudenthal-Tits magic square

The original magic squares [13, 14, 27] are constructions that utilize division algebras in the building of Lie algebras. All the exceptional Lie algebras barring $\mathfrak{g}_2$ appear due to the presence of the octonions in the square. G_2 is, however, the automorphism group of the octonions, so it could be argued that all five exceptional Lie algebras exist because of the largest NDA. The "magic" of the square resides in its symmetry in both input algebras.

Tits' construction of the square uses the degree 3 simple Jordan algebras $\mathfrak{J}(3,\mathbb{A})$ of the NDAs along with derivations. The square is constructed by taking the Jordan algebra of the column algebras ($\mathbb{A}_2$), giving the square's entries by

$$\mathfrak{M}=\mathfrak{der}\,\mathbb{A}_1\oplus\left(\mathbb{A}_1'\otimes\mathfrak{J}'(3,\mathbb{A}_2)\right)\oplus\mathfrak{der}\,\mathfrak{J}(3,\mathbb{A}_2) \tag{2.30}$$

along with defined commutation relations [32]. Here the dash denotes the trace-free part of the algebra. As far as dimensional arguments are concerned, one has as an example $\dim E_8=248=14+(7\times 26)+52$ for the case $\mathbb{A}_1=\mathbb{A}_2=\mathbb{O}$.

Vinberg [33] constructed the square in a manifestly symmetric way by considering instead of Jordan algebras an algebra of traceless antihermitian matrices $\mathfrak{sa}(3,\mathbb{A})$ where

$\mathbb{A} = \mathbb{A}_1 \otimes \mathbb{A}_2$ is a composite algebra of NDAs. By taking the sum of this space and the derivation algebras $\mathfrak{Der}\,\mathbb{A}_{1,2}$ and defining commutation relations between them, a semi-simple Lie algebra arises:

$$\mathfrak{M} = \mathfrak{sa}(3, \mathbb{A}) \oplus \mathfrak{Der}\,\mathbb{A}_1 \oplus \mathfrak{Der}\,\mathbb{A}_2 \tag{2.31}$$

which reduces to the Lie bracket on $\mathfrak{sa}(3, \mathbb{A})$ for the case $\mathbb{A}_{1,2} = \mathbb{R}, \mathbb{C}$.

Another approach, pursued by Barton and Sudbery [6], uses trialities. Using the triality algebra of the input algebras and taking three copies of the tensor product between them, a Lie algebra is defined by

$$\mathfrak{M} = \mathfrak{tri}\,\mathbb{A}_1 \oplus \mathfrak{tri}\,\mathbb{A}_2 \oplus 3(\mathbb{A}_1 \otimes \mathbb{A}_2). \tag{2.32}$$

For the octonionic case, the automorphism group of the triality (see Section 2.4) is Spin(8). The three copies of $\mathbb{A}_1 \otimes \mathbb{A}_2$ are then vector-vector, spinor-spinor and conjugate-conjugate representation tensor products. The NDAs can also be cast into a split form, $\tilde{\mathbb{A}}$, which are no longer division algebras. This is the result of defining certain bases of the algebra to have the property $e_i^2 = 1$, allowing the equation to be 'split'. Building a magic square using the split form of one of the input algebras results in the split real form of the Lie algebras and introduces non-compactness. Thus the algebras that appear are those of groups such as $E_{7(-5)}$.

2.6 Kaluza-Klein reduction of SuGra

It is now generally accepted that the various string theories are manifestations of a larger and more unknown entity designated M-theory, itself 11-dimensional and its lower-energy limit being 11-dimensional supergravity (SuGra). This class of research is currently the best candidate for a unified description of all known fundamental forces, yet if any variations thereof are to describe reality there needs to be a way of reconciling them with the 4-dimensional universe that is observed. The most historically studied method of doing this generalizes ideas developed by Kaluza and Klein [21, 22], where the spatial dimensions are compactified on a d-torus. Compactifications can be done on other manifolds (such as Calabi-Yau manifolds) but they will not be discussed here. Toroidal compactification preserves a maximal amount of super-Poincaré symmetry, $\mathcal{N}$.

2.6.1 Gravity in 5 dimensions

Kaluza-Klein theory was originally proposed to unify the then-known two natural forces of gravity and electromagnetism. In this framework both forces are unified as pure gravity in five dimensions, which results in an Einstein-Maxwell-scalar theory in four dimensions after compactifying one of the original five dimensions on the compact manifold S^1. The 5-dimensional Einstein-Hilbert action is

$$\mathscr{I}_{EH} = \int d^5x \, \hat{\mathscr{R}} \sqrt{-\hat{g}}$$

where $\hat{g} = \det|\hat{g}_{MN}|$ and $\hat{\mathscr{R}}$ is the Ricci scalar (the caret signifying 5 dimensions). All the information in general relativity is encapsulated in the metric. To compactify on the circle of radius L, arbitrarily choose one of the four spatial dimensions, z, and expand the metric as a Fourier series:

$$\hat{g}_{MN}(x^m, z) = \sum_n g_{MN}^{(n)} e^{i\frac{nz}{L}}.$$

Uppercase Latin indices will run over five dimensions. This expansion yields an infinite number of fields in the series, each labelled by Fourier node n. Fields with $n \neq 0$ can be truncated because they correspond to massive fields.

$$\begin{aligned} n = 0 &\quad \text{massless fields} \\ n \neq 0 &\quad \text{massive fields.} \end{aligned}$$

As an illustration consider dimensionally reducing a scalar field from five dimensions to four in this manner.

$$\hat{\phi}(x^m, z) = \sum_n \phi_n(x^m) e^{i\frac{nz}{L}}.$$

The higher-dimensional field obeys the Klein-Gordon equation, $\hat{\Box}\,\hat{\phi} = 0$, but the lower-dimensional right-hand side obeys $\Box\,\phi_n - (n^2/L^2)\phi_n$, the equivalent equation for a scalar of mass $|n/L|$. But taken to the limit $L \to l_p$, i.e., taking S^1 to radius of order Planck length, the mass of the particle becomes so large that expectations of observing it are obliterated. Nonzero Fourier modes are then just truncated to leave the massless sector of the theory. An alternative justification is that reduction on a circle gives $e^{i\frac{nz}{L}}$ which is a group rep of U(1). The rep with $n = 0$ is a singlet and that with $n \neq 0$ is a doublet, implying that n and $-n$ are conjugate representations. If n is then like a U(1) charge there consequently exists a charge conservation law which provides consistency to the truncation of nonzero modes. The ansatz is in effect to ignore z-dependence in the metric:

$$\begin{aligned} \hat{g}_{MN} &\to g_{mn} + g_{mz} + g_{zz} \\ &\to g_{mn} + A_m + \phi, \end{aligned}$$

equivalent to partitioning the metric,

$$\hat{g}_{MN} = \begin{bmatrix} g_{mn} & A_m \\ A_m & \phi \end{bmatrix}.$$

A more natural parameterization which respects symmetries and one which leads to tidy equations of motions is [26]

$$\begin{aligned} \hat{g}_{mn} &= e^{2\alpha\phi} g_{mn} + e^{2\beta\phi} A_m A_n \\ \hat{g}_{nz} &= e^{2\beta\phi} A_n \\ \hat{g}_{zz} &= e^{2\beta\phi} \end{aligned}$$

with α, β tunable parameters. Dimensional reduction of the metric from S^1 compactification yields the metric in the lower dimension, plus a Kaluza-Klein gauge potential and dilaton scalar field, or 'graviphoton' and 'graviscalar', respectively. (Kaluza interpreted this latter field as a 'negative gravitational potential', but did not further ponder its meaning [3]). In compactifying from five to four dimensions this procedure can be identified as reformulating 4-dimensional gravity, Maxwell electromagnetism and dilaton field as pure 5-dimensional Einstein gravity.

2.6.2 M-theory on the torus

String and M-theory naturally incorporate Kaluza-Klein theory, and the same machinery is utilized in compactifying dimensions on successive circles, or tori. The supergravity multiplet contains not just the metric tensor g_{mn} (the graviton in particle theory) but also its superpartner, the gravitino ψ^{α}_{m}, and an additional bosonic 3-form field, C_{mnr}. These will require different considerations for dimensional reduction. Successive compactifications cause a proliferation of fields in the lower-dimensional perspective. A p-form reduced from $N+1$ to N dimensions will effectively result in a p-form and a $(p-1)$-form. In short, forms of all ranks $k \leq p$ appear through successive compactifications. The 128 degrees of freedom of the 11-dimensional spin-$\frac{3}{2}$ gravitino (which has both a vector and a spinor index) decompose into right- and left-handed gravitini and right- and left-handed spinors in 10 dimensions. These continue to decompose into gravitini and spinors in lower dimensions, the details dependent on whether the dimension is odd or even. Scalars simply decompose to scalars, and their production is an important focal point: they contribute to the global symmetry of the theory. (The gravitino will be neglected in this discussion since scalars are bosonic and thus cannot be produced from the dimensional reduction of fermionic fields.)

Compactification of the graviton field on S^1 will give the graviton, vector, and dilaton as discussed above but living in 10 dimensions. The 3-form splits into a 3-form and 2-form in 10 dimensions. Compactifying again on S^1 is the same as compactifying the 11-dimensional theory on the torus, since $T^2 = S^1 \times S^1$. Now there will be 3-, 2- and 1-forms, a graviton, 2 dilatons, 2 vectors, and an axion—a scalar field χ from the decomposition

$$A_m \rightarrow A_m + \chi.$$

The names 'dilaton' and 'axion' are now used differentiate between the scalars arising from the graviton and those arising from the Kaluza-Klein potential or a 1-form field (which will decompose into a 1-form and 0-form, i.e., scalar).

2.6.3 Global symmetries and U-dualities

Gravity on S^1 has the following general covariance (which it inherits from the higher-dimensional gravity) and gauge symmetries:

$$\begin{aligned} \hat{\xi}^M(x,z) &= \xi^m(x) \\ \hat{\xi}^z(x,z) &= cz + \lambda(x) \end{aligned} \tag{2.33}$$

with ξ^M the infinitesimal transformation parameter of the higher-dimensional general covariance and c a constant. Continuing the process of dimensionally reducing the graviton down from $D = 10+1$ to $D = 7+1$, each ith reduction on S^1 gives A^i_m Kaluza-Klein vector potentials. Equation (2.33) is then generalized to

$$\begin{aligned} \hat{\xi}^m(x,z) &= \xi^m(x) \\ \hat{\xi}^i(x,z) &= \Lambda^i_j z^j + \lambda^i(x). \end{aligned} \tag{2.34}$$

Since Λ^i_j is now an object with d^2 components it is simply the group of linear $d \times d$ matrices $\mathrm{GL}(d,\mathbb{R})$. Thus reducing a theory with an Einstein-Hilbert action on T^d gives a theory with what seems to be an additional $\mathrm{GL}(d,\mathbb{R})$ symmetry as well as the usual local co-ordinate and gauge symmetries generated respectively by $\xi^m(x)$ and $\lambda^i(x)$. Such an additional symmetry is a general feature of torus-reduced gravity-coupled-to-matter fields. The actual symmetry is $\mathrm{SL}(d,\mathbb{R})$ unless there is an extra scaling symmetry present in the higher-dimensional equations of motion: $\mathrm{GL}(d,\mathbb{R}) \cong \mathrm{SL}(d,\mathbb{R}) \times \mathbb{R}$. In actual fact the additional symmetry is typically larger than this due to the influence on the global symmetry of the 3-form field decomposition once it is reduced to 8 dimensions.

In the T^2 case there are 2 dilatons, ϕ_i and 1 axion, χ. Combine the dilaton fields into a single field $\vec{\phi} \in \mathbb{C}$ and rotate it such that

$$\phi = a\phi_1 + b\phi_2$$

$$\varphi = a\phi_2 - b\phi_1$$

where a, b are functions of the dimension. In this way, ϕ parameterizes the shape change of the torus (it scales the length of the two circles) while φ parameterizes the torus volume.

Together, ϕ and χ completely characterize the moduli of the torus. Altogether there are 3 scalars with the scalar Lagrangian

$$\mathscr{L} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2. \tag{2.35}$$

Now φ is decoupled from the other two fields: it has a global shift $\varphi + k$ translation symmetry, generating an $\mathbb{R}$ factor in the overall symmetry group. Additionally by defining $\tau = \chi + ie^{-\phi}$ it can be shown that the scalar Lagrangian for ϕ and χ is invariant under

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad ad - bc = 1$$

which is equivalent to Lagrangian invariance under the special linear group SL$(2,\mathbb{R})$. Both Kaluza-Klein potentials transform at once as a doublet under the symmetry group. (In general, the extension of the symmetry group to the rest of the Lagrangian is "guaranteed" once the symmetry of the scalars has been demonstrated. The scalars will always transform non-linearly, and the higher forms as linear representations of the symmetry group.)

The group SL$(2,\mathbb{R})$ is the non-compact version of SU(2); they share the same Lie algebra: $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(2)$. Exponentiating its Cartan subalgebra and the rest of the positive roots reveals a way of writing the Lagrangian using the trace of upper-triangular matrices. In order to keep the upper-triangular form, however, a compensating local orthogonal transformation must be made [29]. The global symmetry group remains the same but the scalar manifold is given by the coset

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{O}(2)} \times \mathbb{R}$$

for the T^2 case. The dimension of this coset is $(3-1)+1 = 3$ i.e., the number of scalars in the theory.

Proceeding to the T^3 case, the decomposition of the graviton and 3-form is given in compact form by employing Dynkin notation:

$$\begin{aligned} g_{mn} \quad & [2000]_9 \to [200]_6 + 3[100]_6 + 3\phi + 3\chi \\ B_{mnr} \quad & [0010]_9 \to [001]_6 + 3[010]_6 + 2[100]_6 + \chi. \end{aligned} \tag{2.36}$$

Since $d = 3$ one could expect a naive global symmetry of $GL(3,\mathbb{R})$ for the scalar Lagrangian, and therefore a naive coset manifold of $GL(3,\mathbb{R})/O(3)$ just from the graviton alone. The dimension of the coset is $9-3 = 6$, the number of scalars from the graviton and Kaluza-Klein potentials. However, the actual symmetry group must be larger. (One suspects it is 7-dimensional, since seven scalars have been produced.) Actually one of the dilatons, as before, is decoupled from the rest of the graviton scalars and couples with the axion of the 3-form field. Thus one actually has a coset from each coupling:

$$\frac{SL(3,\mathbb{R})}{O(3)} \times \frac{SL(2,\mathbb{R})}{O(2)}$$

with dimension $(8-3)+(3-1) = 7$. The pattern emerging is that of a scalar coset manifold G_d/H_d where G_d is the Lie algebra of the global symmetry group and H_d is its maximal compact subgroup.

For compactifications as far as $D = 6$ the dilatons are just arranged as vectors corresponding to the positive roots of the algebra. To continue to lower dimensions one needs to know which roots are simple, i.e., which subset of dilaton vectors can express all others. Dimensional reduction on T^d only continues to $D = 3$ where the graviton and gravitini

D	T^d	G_d/H_d	no. scalars
10	T^1	$\frac{SL(2,\mathbb{R})}{O(2)}$	$3-1=2$
9	T^2	$\frac{SL(2,\mathbb{R})}{O(2)} \times \mathbb{R}$	$(3-1)+1=3$
8	T^3	$\frac{SL(3,\mathbb{R})}{O(3)} \times \frac{SL(2,\mathbb{R})}{O(2)}$	$(8-3)+(3-1)=7$
7	T^4	$\frac{SL(5,\mathbb{R})}{O(5)}$	$24-10=14$
6	T^5	$\frac{SO(5,5)}{O(5)\times O(5)}$	$45-(10+10)=25$
5	T^6	$\frac{E_{6(6)}}{USp(8)}$	$78-36=42$
4	T^7	$\frac{E_{7(7)}}{SU(8)}$	$133-63=70$
3	T^8	$\frac{E_{8(8)}}{O(16)}$	$248-120=128$

Table 2.2: The coset manifolds of SuGra on T^d.

have zero degrees of freedom. In lower dimensions gravity becomes a constraint.

It can be seen that dimensionally-reduced SuGra displays non-compact global symmetry of $\mathrm{E}_{d(d)}$ type. For instance, $G_4 = \mathrm{SL}(5,\mathbb{R})$ which has Lie algebra $\mathfrak{a}_4 \cong \mathfrak{e}_4$. This has been known for some time [7, 8, 20]. The group G_d is actually the U-duality group: the non-perturbative symmetry combining the transformations of S-duality and T-duality [25].

OCTONIONIC SUPER YANG-MILLS

It is known that super Yang-Mills (SYM) fields can only exist in dimensions 3,4,6 and 10. The reason for this is actually dependent on the existence of a normed division algebra of dimension $n = D-2$. In fact they are mutually implicative. The properties of the NDAs cause the vanishing of a certain trilinear spinor term that makes the SYM Lagrangian of a nonabelian Yang-Mills field minimally coupled to a massless spinor supersymmetric in the aforementioned spacetime dimensions [5]. They are also responsible for the supersymmetry of the Green-Schwarz superstring [15]. The purpose of this chapter is to demonstrate the dimensional reduction of SYM merely using the NDA over which the theory is constructed.

Once formulated in the octonions, $D = 10$ SYM theory can be reduced to obtain the $D = 6,4,3$ SYM in the other NDAs. The 10-dimensional SYM has $\mathcal{N} = 1$, consisting of vector (1-form) and spinor superpartners. Since the spacetime dimension is $D = 10$ the fields can be formulated as in the Little group as single octonions (see Section 2.3),

$$\psi_{\mathbb{O}} = \psi^{\mu} e_{\mu}$$
$$v_{\mathbb{O}} = v^{\mu} e_{\mu},$$

with components transforming with the $\mathbf{8}_s$ and $\mathbf{8}_v$ of SO(8) of Section 2.2:

$$\delta\psi^{\mu} = \frac{1}{2}\theta^{\rho\sigma}\Sigma^{[\rho\sigma]}_{\mu\nu}\psi^{\nu} \tag{3.1}$$
$$\delta v^{\mu} = \frac{1}{2}\theta^{\rho\sigma}J_{[\rho\sigma]\,\mu\nu}v^{\nu}. \tag{3.2}$$

This can be seen since application of (2.25) gives:

$$\begin{aligned}\delta\psi_{\mathbb{A}} &= \frac{1}{4}\theta^{\mu\nu}e^{*}_{\mu}(e_{\nu}\psi_{\mathbb{A}})\\ &= \frac{1}{4}\theta^{\mu\nu}e^{*}_{\mu}(e_{\nu}e_{\rho})\psi^{\rho}\\ &= \frac{1}{4}\theta^{\mu\nu}e^{*}_{\mu}e_{\sigma}\Gamma^{\nu}_{\rho\sigma}\psi^{\rho}\\ &= \frac{1}{4}\theta^{\mu\nu}\bar{\Gamma}^{\mu}_{\sigma\tau}\Gamma^{\nu}_{\rho\sigma}\psi^{\rho}e_{\tau}\\ &= \frac{1}{2}\theta^{\mu\nu}\Sigma^{[\mu\nu]}_{\tau\rho}\psi^{\rho}e_{\tau}\end{aligned}$$

with a relabeling of indices to give (3.1) and

$$\begin{aligned}\delta v_{\mathbb{A}} &= \frac{1}{4}\theta^{\mu\nu}\Big(e_{\mu}(e^{*}_{\nu}v_{\mathbb{A}}) - v_{\mathbb{A}}(e^{*}_{\mu}e_{\nu})\Big)\\ &= \frac{1}{4}\theta^{\mu\nu}\Big(e_{\mu}(e^{*}_{\nu}e_{\rho}) - e_{\rho}(e^{*}_{\mu}e_{\nu})\Big)v^{\rho}\\ &= \frac{1}{4}\theta^{\mu\nu}\Big(e_{\mu}e_{\sigma}\bar{\Gamma}^{\nu}_{\rho\sigma} - e_{\rho}e_{\sigma}\bar{\Gamma}^{\mu}_{\nu\rho}\Big)v^{\rho}\\ &= \frac{1}{4}\theta^{\mu\nu}\Big(\Gamma^{\mu}_{\sigma\tau}\bar{\Gamma}^{\nu}_{\rho\sigma} - \Gamma^{\rho}_{\sigma\tau}\bar{\Gamma}^{\mu}_{\nu\rho}\Big)v^{\rho}e_{\tau}\\ &= \frac{1}{2}\theta^{\mu\nu}\Big(\delta_{\tau\mu}\delta_{\nu\rho} - \delta_{\tau\nu}\delta_{\mu\rho}\Big)v^{\rho}e_{\tau}\\ &= \frac{1}{2}\theta^{\mu\nu}J_{[\mu\nu]\tau\rho}v^{\rho}e_{\tau}\end{aligned}$$

with the Clifford condition and relabeling to give (3.2). Once $D = 10, \mathcal{N} = 1$ SYM has be given in terms of octonions, dimensional reduction on T^4 (the 4-torus) gives the octonionic $D = 6, \mathcal{N} = 2$ theory. Obtaining the $\mathcal{N} = 1$ theory is then merely a matter of truncation. One is then left with $D = 6, \mathcal{N} = 1$ SYM over the quaternions.

3.1 SYM on T^4

From Kaluza-Klein reduction (see Section 2.6) the SO(8) Little vector $v_{\mathbb{O}}$ representation in $D = 10$ will reduce to a single SO(4) ≅ SU(2)×SU(2) vector plus 4 scalar fields transforming in the internal symmetry group (which in the T^4 case is SO(4) also). (It should be noted that the internal symmetry is *not* the same as the R-symmetry; in particular cases the former is larger than the latter.) The spinor representation reduces to the spinor and conjugate spinor representations of $SU(2)\times SU(2)$. The overall decomposition leads to a spacetime part and an internal part:

$$\begin{aligned} &\mathrm{SO}(8) \supset \mathrm{SO}(4)_{ST} \times \mathrm{SO}(4)_I \\ \text{with} \quad &\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2). \end{aligned} \tag{3.3}$$

The vector and spinors are quaternionic while only the scalars are real. The octonion corresponding to the spinor decomposes in the opposite way it was constructed by two elements of $\mathbb{H}$ and an arbitrary imaginary element, i, in the Cayley-Dixon procedure—'Dixon undoubling': $\psi_{\mathbb{O}} \to \psi_{\mathbb{H}} + i\chi_{\mathbb{H}}$. The real part plus a line from the Fano plane comprise the spinor subfield, leaving the remaining quadrangle representing the compactified directions. The imaginary element factorizes these bases into a quaternionic conjugate spinor subfield, with the same basis a the $D = 6$ spinor. The vector is deconstructed by removing an arbitrary line of the Fano plane to form a quaternionic vector with the remaining four imaginary bases parameterizing four real scalars: $v_{\mathbb{O}} \to v_{\mathbb{H}} + \overrightarrow{\phi}_{\mathbb{R}}$. The internal co-ordinates are denoted by special indices: $\overrightarrow{\phi}_{\mathbb{R}} = \phi^{\dot{\mu}} e_{\dot{\mu}}$, where in this case both dotted and undotted indices can take four values. Internal co-ordinates correspond to the parameterization of the torus on which the theory is compactified. Correspondingly, $\theta^{\mu\nu}$ decomposes into spacetime and internal parts, $\theta^{\mu\nu}$ and $\theta^{\dot{\mu}\dot{\nu}}$, while $\theta^{\mu\dot{\mu}} = 0$.

One has the vector, spinors and scalar representations of $SU(2)\times SU(2)$ written over the octonions. This is the $\mathcal{N} = 2$ SYM theory in $D = 6$. Anastasiou et al. [1] demonstrate an explicit decomposition of the octonionic fields in addition to their internal and spacetime transformations. They are as follows:

$$\delta^{(ST)}\psi_{\mathbb{H}} = \theta^{ST}_{\mathbb{H}}\psi_{\mathbb{H}} \tag{3.4}$$

$$\delta^{(ST)}\chi_{\mathbb{H}} = \tilde{\theta}^{ST}_{\mathbb{H}}\chi_{\mathbb{H}} \tag{3.5}$$

$$\delta^{(ST)}v_{\mathbb{H}} = \tilde{\theta}^{ST}_{\mathbb{H}}v_{\mathbb{H}} - v_{\mathbb{H}}\theta^{ST}_{\mathbb{H}} \tag{3.6}$$

$$\delta^{(ST)}\phi_{\mathbb{H}} = 0 \tag{3.7}$$

for spacetime transformations, and

$$\delta^{(I)}\psi_{\mathbb{H}} = \psi_{\mathbb{H}}\theta^I_{\mathbb{H}} \tag{3.8}$$

$$\delta^{(I)}\chi_{\mathbb{H}} = \chi_{\mathbb{H}}\tilde{\theta}^I_{\mathbb{H}} \tag{3.9}$$

$$\delta^{(I)}v_{\mathbb{H}} = 0 \tag{3.10}$$

$$\delta^{(I)}\phi_{\mathbb{H}} = \tilde{\theta}^I_{\mathbb{H}}\phi_{\mathbb{H}} - \phi_{\mathbb{H}}\theta^I_{\mathbb{H}} \tag{3.11}$$

for internal transformations. Since the bases are quaternionic the following definitions have been made:

$$\theta^{ST}_{\mathbb{H}} \equiv \frac{1}{4}\theta^{\mu\nu}(e^*_\mu e_\nu)\,, \qquad \tilde{\theta}^{ST}_{\mathbb{H}} \equiv \frac{1}{4}\theta^{\mu\nu}(e_\mu e^*_\nu)$$

with $\theta^I_{\mathbb{H}}$ and $\tilde{\theta}^I_{\mathbb{H}}$ defined similarly.

3.2 SYM on T^6

Compactifying SYM in $D = 10$ now to SYM in $D = 4$. The Little group decomposes according to

$$\begin{aligned} &\mathrm{SO}(8) \supset \mathrm{SO}(2)_{ST} \times \mathrm{SU}(4)_I \\ &\mathrm{SO}(2) \cong \mathrm{U}(1)\,, \quad \mathrm{SO}(6) \cong \mathrm{SU}(4) \end{aligned} \tag{3.12}$$

so the vector rep of SO(8) splits into a reducible vector rep of SO(6) and 6 scalar fields, while, since the D_3 Dynkin diagram is symmetric, the spinors fall into copies of the spinor and conjugate spinor reps of SU(4) distinguished by their charge under U(1). The octonionic fields are deconstructed in a similar manner as in the previous section. The spinor fields are now complex, and can be found by choosing an arbitrary line of the Fano plane and using it to arrange the octonionic field as 4 complex fields: $\psi_{\mathbb{O}} \to \psi_{\mathbb{R}} + i\bar{\psi}_{\mathbb{R}} + j\chi_{\mathbb{R}} + k\bar{\chi}_{\mathbb{R}}$ with the bar denoting conjugacy. One of the remaining imaginary bases along with the real basis is chosen from the Fano plane to construct a complex vector and 6 real scalars from an octonion: $v_{\mathbb{O}} \to v_{\mathbb{R}} + \vec{\phi}_{\mathbb{R}}$ where the dotted indices of $\vec{\phi}_{\mathbb{R}} = \phi^{\dot{\mu}}e_{\dot{\mu}}$ now run over 6 points of the Fano plane. A consequence is that the spacetime transformation parameter $\theta^{\mu\nu}$ is just a single parameter θ. The factor appearing with it in the transformations is the U(1) charge, which is $\pm\frac{1}{2}$ for spinors, ± 1 for vectors and 0 for scalars. The spacetime transformations are

$$\delta^{(ST)}\psi_{\mathbb{C}} = \frac{1}{2}\theta\psi_{\mathbb{H}} \tag{3.13}$$

$$\delta^{(ST)}\bar{\psi}_{\mathbb{C}} = \frac{1}{2}\theta\bar{\psi}_{\mathbb{H}} \tag{3.14}$$

$$\delta^{(ST)}\chi_{\mathbb{C}} = -\frac{1}{2}\theta\chi_{\mathbb{H}} \tag{3.15}$$

$$\delta^{(ST)}\bar{\chi}_{\mathbb{C}} = -\frac{1}{2}\theta\bar{\chi}_{\mathbb{H}} \tag{3.16}$$

$$\delta^{(ST)}v_{\mathbb{C}} = -\theta i v_{\mathbb{C}} \tag{3.17}$$

$$\delta^{(ST)}\phi_{\mathbb{R}} = 0 \tag{3.18}$$

where i is an arbitrary imaginary octonionic base. The spinors can also be arranged to form a quadruplet. This then can be shown [1] to transform under the fundamental rep of SU(4), such that

$$\delta^{(I)}\psi^{\mu} = T^{[\dot{\mu}\dot{\nu}]}_{\mu\nu}\psi^{\nu} \tag{3.19}$$

with $T^{[\dot{\mu}\dot{\nu}]}$ the generator of SU(4). The vector again transforms trivially (i.e., as a singlet), this time under SU(4). The arrangement of the 6 scalars transform under the 6-dimensional vector representation.

$$\delta^{(I)}v_{\mathbb{C}} = 0 \tag{3.20}$$

$$\delta^{(I)}\phi^{\dot{\rho}} = \frac{1}{2}\theta^{\dot{\mu}\dot{\nu}}J_{[\dot{\mu}\dot{\nu}]\dot{\rho}\dot{\sigma}}\phi^{\dot{\sigma}} \tag{3.21}$$

All the generators have been built with the multiplication rule of the octonions. One therefore has the $\mathcal{N}=4$ SYM formulated over $\mathbb{O}$. As before it is a matter of truncation to arrive at $\mathcal{N}=2$: by discarding 4 points of the Fano plane one is left with a line corresponding to the $\mathcal{N}=2$ theory over $\mathbb{H}$, translating to the throwing away of the spinor conjugates $\bar{\psi}_{\mathbb{C}}, \bar{\chi}_{\mathbb{C}}$ and 4 of the scalars. Further, the $\mathcal{N}=1$ theory can be reached by removing two more imaginary bases, leaving SYM over $\mathbb{C}$. This translates to discarding the conjugate spinor $\chi_{\mathbb{C}}$ and the two remaining scalar fields. In all cases, truncation leaves an equal number of bosonic and fermionic degrees of freedom to preserve SuSy.

3.3 SYM on T^7

Preceding to $D=3$, a consequence of the reduction is that there is no spacetime symmetry. The decomposition follows

$$\mathrm{SO}(8) \supset \mathrm{SO}(7)_I \tag{3.22}$$

The $D=10$ spinor field decomposes into 8 spinor fields in the $D=3$ theory: $\psi_{\mathbb{O}} \to \psi^{\mu}_{\mathbb{R}} e_\mu$. The vector reduces to a trivial real vector with 7 scalar fields: $v_{\mathbb{O}} \to v_{\mathbb{R}} + \overrightarrow{\phi}_{\mathbb{R}}$, where the notation is now reverted to that of Section 2.1 since the imaginary subspace of an octonion now corresponds to the internal symmetry: $\overrightarrow{\phi}_{\mathbb{R}} = \phi^{\dot{\mu}} e_{\dot{\mu}} = \phi^i e_i$. The vector is now just the real component of the original $D=10$ vector octonion: $v_{\mathbb{R}} = v_0 1$. All fields are now real.

The spinors transform as the **8** of SO(7) because the transformation parameters θ^{ij} build the spin generators of Section 2.2:

$$\begin{aligned} \delta^{(I)} \psi_{\mathbb{O}} &= \frac{1}{4}\theta^{ij} e_i^* (e_j e_\mu)\psi^\mu = -\frac{1}{2} e_\nu \Sigma^{[ij]}_{\nu\mu} \psi^\mu \\ \delta^{(I)} \psi^\mu &= -\frac{1}{2}\Sigma^{[ij]}_{\mu\nu}\psi^\nu . \end{aligned} \tag{3.23}$$

The scalars transform as the **7** of SO(7):

$$\delta^{(I)} \phi^k = \frac{1}{2}\theta^{ij} J_{[ij]kl}\phi^l . \tag{3.24}$$

Dimensional reduction to $D=3$ yields the maximal $\mathcal{N}=8$ SYM theory, and here one has it formulated over $\mathbb{O}$. The same truncation procedure as in the previous sections leads to the $\mathcal{N}=1$, $D=3$ theory over $\mathbb{R}$. In that theory one is left with a superpartner pair made up of a real vector and real spinor.

In $D=3$ the vector is dual to a scalar field. Thus the SO(7) internal symmetry in enlarged to SO(8), the R-symmetry group of the theory. (Note that for $\mathcal{N}=2,3$ the internal symmetry is larger than the R-symmetry.) The $D=3, \mathcal{N}=8$ SYM Lagrangian is given by

$$\begin{aligned} \mathscr{L} = \frac{1}{4} F^A_{mn} F^{Amn} - \frac{1}{2} D_m \phi^A_i D^m \phi^A_i + i \bar{\lambda}^A_\mu \gamma^m D_m \lambda^A_\mu \\ - \frac{1}{4} g^2 f^A_{BC} f^A_{DE} \phi^B_i \phi^D_i \phi^C_j \phi^E_j - g f^\mu_{BC} \phi^B_i \bar{\lambda}^{A\mu} \Gamma^i_{\mu\nu} \lambda^{C\nu} \end{aligned} \tag{3.25}$$

where one should recognize that $\Gamma^i_{\mu\nu}$ ($i=1,\ldots,7$; $\mu=0,\ldots,7$) belongs to the Clifford algebra of Spin(7) and is an octonionic structure constant. Thus the Lagrangian can be written over $\mathbb{O}$. If $\mathbb{O}$ is replaced by the division algebras $\mathbb{H},\mathbb{C},\mathbb{R}$ then the result is $\mathscr{L}$ with $\mathcal{N}=4,2,1$.

3.4 Division-algebraic classification

In order to build up a unifying picture of SYM and SuGra provided by the NDAs, the theories of the former are tabulated in Table 3.1. Each entry gives the algebra over which the theory is written for the corresponding D and $\mathcal{N}$. It has already been shown that an NDA of dimension n is isomorphic to a spacetime symmetry group of $D = n+2$, but another emergent correspondence is that of the NDAs with the amount of supersymmetry $\mathcal{N}$. Both have been displayed in parentheses with their respective D and $\mathcal{N}$. One sees

$D \setminus \mathcal{N}$	1 ($\mathbb{R}$)	2 ($\mathbb{C}$)	4 ($\mathbb{H}$)	8 ($\mathbb{O}$)
10 ($\mathbb{O}$)	$\mathbb{O}$			
6 ($\mathbb{H}$)	$\mathbb{H}$	$\mathbb{O}$		
4 ($\mathbb{C}$)	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	
3 ($\mathbb{R}$)	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$

Table 3.1: The division algebraic classification of Super Yang-Mills theories.

that when for example $D = 4, \mathcal{N} = 4$ one has $\mathbb{C} \times \mathbb{H} = \mathbb{O}$. Table 3.2 displays the full on-shell symmetry group (that is, without distinguishing between spacetime and R-symmetries) of each theory. A point on which to remark is that although \-shaped diagonal lines of the table have theories formulated over the same algebra, not all have the same symmetry groups. Only when $D = 3$ or $\mathcal{N} = 1$ (i.e., when one of the algebras is $\mathbb{R}$) do the symmetry groups match. The symmetry groups are then clearly not fully determined by the algebra over which the theory is written, but by which of the "input" algebras are spacetime or internally generated. Also of interest is that for $D = 3$ the symmetry groups are those of the triality automorphism groups from Section 2.4 for each NDA. (Why this correspondence only appears for $D = 3$ is a subtler point involving limitations of the triality definition. This will be discussed in the following chapter.) The $D = 3$ symmetry groups are the same as those with $\mathcal{N} = 1$. This then is only natural considering $\mathcal{N} = 1$ SuSy transformations are

$D \setminus \mathcal{N}$	1	2	4	8
10	SO(8)			
6	$\mathrm{sp}(2)^3$	$\mathrm{Sp}(2)^4$		
4	$\mathrm{U}(1)^2$	$\mathrm{Sp}(2)\times\mathrm{U}(1)^3$	$\mathrm{SU}(4)\times\mathrm{U}(1)$	
3	1	$\mathrm{U}(1)^2$	$\mathrm{Sp}(2)^3$	SO(8)

Table 3.2: The full symmetry groups of Super Yang-Mills theories.

of the form

$$v = \chi\psi^*, \qquad \psi = v^*\chi, \qquad \chi = a\psi.$$

But this is just the triality relationship between between the three reps of $\mathrm{SO}(\mathbb{A})$ when written over $\mathbb{A}$ (e.g., $\psi = a^\mu \chi^\rho \Gamma^\mu_{\nu\rho}$) and since the group that preserves this relationship is the triality group it is unsurprising that the full on-shell symmetries of the $D = 3$ theories are exactly the triality algebras of each, inherited from the $\mathcal{N} = 1$ theories [1, 2].

GRAVITY AS A SQUARE OF PARTICLE THEORIES

This chapter introduces the crux of this dissertation, demonstrating how the octonions play a part in the construction of theories of gravity, an updated description of which is a major aspect of certain efforts toward unification, as elaborated in Chapter 1. One sees the development of what seems to be a recurring theme in recent literature. It has been found [31,36–38] that taking double copies—"squaring"—of two-algebra SYM amplitudes yields three-dimensional SuGra amplitudes, adding to the examples [19,28] of gravity appearing as a square of particle physics. Here it is shown that this theme is intimately tied in to the magic square of Lie algebras.

4.1 Tensoring SYM

Two SYM multiplets can be tensored to give the content of a SuGra theory. For instance, in $D = 10$, the tensor product in Dynkin notation could be

$$\begin{aligned}\left([1000]_8 + [0010]_8\right) \otimes \left([1000]_8 + [0001]_8\right)& \\ &= [2000]_{35} + [0011]_{56} + [0100]_{28} + [0000]_1 + [1001]_{56} \\ &\quad + [1010]_{56} + [1000]_8 + [0001]_8 + [0010]_8 \\ &= g_{mn} + C_{mnp} + B_{mn} + \phi + \xi^{\alpha}_{m} + \bar{\xi}^{\dot{\alpha}}_{m} + A_m + \chi^{\alpha} + \bar{\chi}^{\dot{\alpha}}\end{aligned}$$

which is the massless field content of $D = 11$ SuGra reduced on S^1 (i.e., type II). Note that now the subscripts of the Dynkin notation denote the dimension of the irreducible representations, as opposed to that of the Little group. Both sides add to 265—thus the bosonic and fermionic degrees of freedom are separately conserved as they should be. In such a tensoring the number of supersymmetries add, so tensoring two $\mathcal{N} = 16$ SYM multiplets gives an $\mathcal{N} = 32$ SuGra.

The $D = 3$ Lagrangian in (3.25) can be rewritten over NDAs, allowing for SYM with different amounts of supersymmetries. The SYM multiplet then contains a vector and spinor(s) in $\mathbb{A}$ and an $\text{Im}\,\mathbb{A}$-valued scalar field, $\vec{\phi}$. By tensoring these $D = 3$, $\mathcal{N} = 1, 2, 3, 4$ multiplets 16 $D = 3$, $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ SuGra theories are constructed. (The reason only $D = 3$ tensoring is being considered for now is due to the triality algebra restriction alluded to in the previous chapter.) The content of these theories are valued as follows:

$$g_{mn} \in \mathbb{R}\,, \qquad \xi^{\alpha}_{m} \in \begin{bmatrix} \mathbb{A}_1 \\ \mathbb{A}_2 \end{bmatrix}, \qquad \phi, \psi \in \begin{bmatrix} \mathbb{A}_1 \otimes \mathbb{A}_2 \\ \mathbb{A}_1 \otimes \mathbb{A}_2 \end{bmatrix}. \tag{4.1}$$

In $D = 3$ the graviton g_{mn} and gravitino ξ^{α}_{m} have zero degrees of freedom, while the scalar and spinor fields both have $2(n_1 \times n_2)$ degrees of freedom (128 each in the maximal $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{O}$ case, as when 11-dimensional SuGra is reduced on the 8-torus). The obtained SuGra theories with $\mathcal{N} > 8$ are unique and all content lives in the SuGra multiplet, whereas those with $\mathcal{N} \leq 8$ can be coupled to additional matter multiplets. Each theory has its own U-duality and scalar coset G/H.

4.2 Magic square of $D = 3$ SuGras

Borsten et al. [24] have shown that one can arrive at the $D = 3$ SuGra theories, introduced above as tensored SYM multiplets, from a magic square construction (Section 2.5). The Barton and Sudbery magic square construction uses triality algebras and introduces non-compactness in the group algebras it produces.

$$\mathfrak{M} = \mathfrak{tri}\,\mathbb{A}_1 \oplus \mathfrak{tri}\,\mathbb{A}_2 \oplus 3(\mathbb{A}_1 \otimes \mathbb{A}_2). \tag{4.2}$$

By adapting the commutators of the Barton-Sudbery construction Borsten et al. [24] have used (4.2) to construct the magic square of non-compact Lie algebras in Table 4.1, providing a more manifestly $\mathbb{A}_1 \leftrightarrow \mathbb{A}_2$ symmetric square. What is remarkable about this magic

$\mathbb{A}_1 \setminus \mathbb{A}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(2,1)$	$\mathfrak{sp}(4,2)$	$\mathfrak{f}_{4(-20)}$
$\mathbb{C}$	$\mathfrak{su}(2,1)$	$\mathfrak{su}(2,1)^2$	$\mathfrak{su}(4,2)$	$\mathfrak{e}_{6(-14)}$
$\mathbb{H}$	$\mathfrak{sp}(4,2)$	$\mathfrak{su}(4,2)$	$\mathfrak{so}(8,4)$	$\mathfrak{e}_{7(-5)}$
$\mathbb{O}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(8)}$

Table 4.1: Magic square of Lie algebras corresponding to the U-dualities of $D = 3$ Supergravities.

$\mathbb{A}_1 \setminus \mathbb{A}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$	$\mathfrak{so}(9)$
$\mathbb{C}$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(2)^2 \oplus \mathfrak{u}(1)^2$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$
$\mathbb{H}$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4)$	$\mathfrak{so}(12) \oplus \mathfrak{so}(3)$
$\mathbb{O}$	$\mathfrak{so}(9)$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$	$\mathfrak{so}(12) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(16)$

Table 4.2: Magic square of maximal compact subalgebras.

square is that the entries are the U-duality groups of the $D = 3$ supergravities introduced in the previous section. Furthermore, Table 4.2 is the magic square of maximal compact subalgebras obtained by the *reduced* triality construction,

$$\mathfrak{M}' = \mathfrak{tri}\, \mathbb{A}_1 \oplus \mathfrak{tri}\, \mathbb{A}_2 \oplus (\mathbb{A}_1 \otimes \mathbb{A}_2). \tag{4.3}$$

One can see that for the case $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{O}$ the U-duality group and maximal compact subgroup give the coset

$$\frac{G_8}{H_8} = \frac{E_{8(8)}}{\mathrm{SO}(8)}$$

which is that of 11-dimensional supergravity reduced to $D = 3$ on the 8-torus (Subsection 2.6.3). In fact it is readily seen that, since in the cases $D = 3$ their symmetries are actually the trialities, tensoring SYM multiplets will yield supergravities with symmetries that are the sum of trialities in the fashion of (4.2). That these deliver the precise U-dualities for SuGra multiplets and SuGra multiplets coupled to extra matter fields is another example of gravity appearing as a square of particle physics—in this instance as a magic square. Hence a connection between the magic square of algebras in mathematics and the SuGra square of SYM in physics has been identified.

4.3 A pyramid of gravitation

The definition of $\mathfrak{tri}\,\mathbb{A}$ used in (2.27) in not inclusive enough to give the full on-shell symmetry groups of all the SYM theories except for those with $D = 3$ or $\mathcal{N} = 1$. It has already been shown that a given SYM theory is not just dependent on the algebra over which it is written but on the two algebras corresponding to D and $\mathcal{N}$. Thus it seems fitting to extend the triality algebra to depend on both of these, which will be defined as $\mathbb{A}_n$ and $\mathbb{A}_{\mathcal{N}}$. If such a triality algebra $\tilde{\mathfrak{tri}}(\mathbb{A}_n, \mathbb{A}_{\mathcal{N}})$ can be constructed then the magic square can be generalized to supergravities in dimensions $n+2$, adding a new axis to the square.

It transpires that the part of the triality algebra of (2.27) that is restricted during dimensional reduction is the subalgebra corresponding to the spacetime symmetry [1]. Since the vector rep always transforms in this symmetry and is an element of the $\mathbb{A}_n$ subalgebra of $(\mathbb{A}_n, \mathbb{A}_{\mathcal{N}})$, an additional condition that $\mathbb{A}_n$ is preserved by any element of the triality triple A, B, C can be included to define

$$\begin{gathered}\tilde{\mathfrak{tri}}\,\mathbb{A} \equiv \left\{(A,B,C)|A(xy) = (Bx)y + x(Cy) \text{ and } A(\mathbb{A}_n \subseteq (\mathbb{A}_n, \mathbb{A}_{\mathcal{N}})) = \mathbb{A}_n\right\} \\ \text{with } A,B,C \in \mathfrak{so}(\mathbb{A}), \quad x,y \in (\mathbb{A}_n, \mathbb{A}_{\mathcal{N}}).\end{gathered} \tag{4.4}$$

This updated triality algebra now gives the full on-shell symmetry group for any SYM in $D = n+2$. Recalling from Chapter 3 that reducing octonion SYM theories on the d-torus induces extended supersymmetry which itself can be truncated, the obvious constraint placed on the theories of symmetries that can be built from (4.4) is $\dim\mathbb{A}_n + \dim\mathbb{A}_{\mathcal{N}} \leq 9$. The result of squaring these is then a stepped magic pyramid of supergravities [2, 24] in $n+2$ dimensions, with the magic square as its base. One should note that the $\mathbb{A}_n = \mathbb{A}_{\mathcal{N}} = \mathbb{O}$

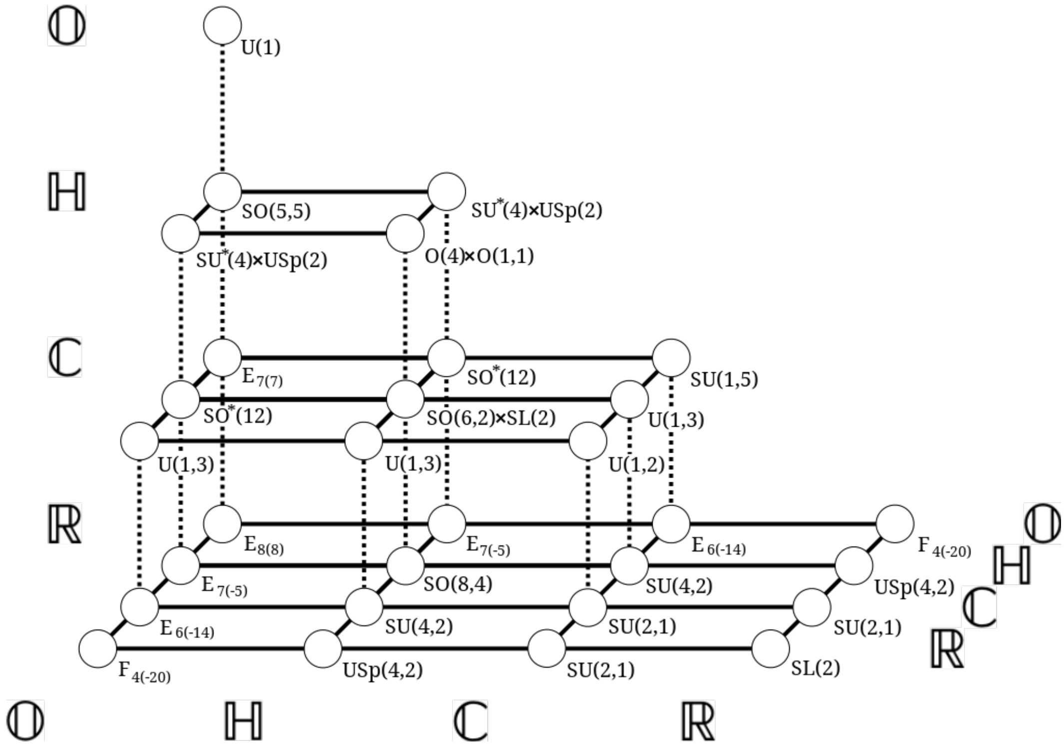

Figure 4.1: The magic pyramid of supergravity U-duality groups labelled by NDAs.

corner of the pyramid displays the symmetry groups of toroidal compactifications of 11-supergravity to D =10, 6, 4 and 3.

The capstone of the pyramid is type II supergravity, the low-energy limit of type II string theory (11-dimensional SuGra on S^1). The magic pyramid is still in the early days of research, its importance and implications either yet to be unearthed or (at time of writing) published. But a possible assignment is that it represents a description of the structure of SuGra symmetries via the the normed division algebras, labelling each theory using three of them, and placing the octonions in a prominent position as corresponding to the 10-dimensional type II symmetry.

CLOSING REMARKS

The octonions have found little accommodation in mainstream physics, in part due to their non-associativity. The work presented above demonstrates that their appearance as the last and largest normed division algebra is not only to be expected, but possibly to have significance of its own. But though the existence of the exceptional Lie algebras can be ascribed to the presence of octonions, their ultimate relevance to physics remains inconclusive.

The octonionic aspect of SYM theories is already known, with the existence of the NDAs providing their supersymmetry. Their link with higher-dimensional spacetimes heavily relies on exploiting the Lie algebra isomorphism between $\mathrm{SO}(1, n+1)$ Lorentz symmetries of dimension $D = n+2$ and the 2×2 special linear matrices with entries in $\mathbb{A}$, which allows the generalization of the Pauli and gamma matrices. Starting from these relationships and tensoring SYM multiplets, a connection between the division algebras and supergravity in $n+2$ dimensions has been uncovered through the magic pyramid, itself a contribution to pure mathematics through a refinement of the triality algebra definition. This in turn has been built from the hitherto purely mathematical magic square of Lie algebras after considering a degeneracy in the SYM theories. These remarkable structures, now of nascent relevance to physical ideas, provide the U-duality (symmetry) groups for known higher-dimensional supersymmetric theories of gravity. Concomitant with the magic structures' involvement herein are the triality and Jordan algebras with which they can be constructed. The latter of these is related to the question of whether or not octonionic quantum mechanics exists.

Formulating SYM over the octonions allows a unique dimensional reduction in the fashion of Dixon undoubling, corresponding to reduction on the d-torus. In the $(\mathbb{O},\mathbb{O})$ corner of the magic pyramid one sees the U-duality groups of Kaluza-Klein-type toroidal compactification of SuGra. A logical step is to formulate the SuGra fields themselves octonionically using the $SO(1, n+2)$ extension of the above isomorphism, and then to decompose the fields directly. A further line of research will be to clarify how dimensional reduction of SuGra on other manifolds relates to the decomposition of octonionic fields in the context of the pyramid. It may be interesting to look at G_2-invariant compactifications or at manifolds of G_2 type, for this group is the automorphism group of $\mathbb{O}$.

The division algebraic formulation of SYM at the level of the Lagrangian has been mostly omitted from the present discussion, though it has been studied [1]. Further research to illuminate the "SuGra = SYM^2" relation at the Lagrangian level remains to be carried out. In order to do this the relationships between the fields of each theory will need to be shown, as well as clearly demonstrating how the SuGra transformation rules emerge as a consequence of those of SYM.

In short, the unifying picture provided by the normed division algebras is striking and gives ample motivation to continue their study in the context of SuSy and SuGra. It is hoped this picture will present the octonions with their first physical manifestation: as an algebraic aspect of supersymmetric gravity for studying a contender theory of all known interactions.

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